Math 246B Lecture 3 Notes

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1 Local Uniform Convergence, Upper Semicontinuity, and Subharmonic Functions

1.1 Local uniform convergence of harmonic functions

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $u \in C(\Omega)$ be such that for all $a \in \Omega$, there exists $R_n \to 0$ such that*

$$u(a) = \frac{1}{2\pi R_n} \int_{|y|=R_n} u(a+y)\, ds(y)$$

for all n. Then $u \in H(\Omega)$.

Corollary 1.1. *Let $u_j \in H(\Omega)$ be a sequence such that $u_k \to u$ locally uniformly in Ω. Then $u \in H(\Omega)$, and for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we have $\partial^\alpha u_k \to \partial^\alpha u$ locally uniformly in Ω. Here, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$.*

Proof. By the theorem, u has the mean value property, so $u \in H(\Omega)$. If $\{|x-a| \leq R\} \subseteq \Omega$, write (for $|x-a| \leq R/2$)

$$\begin{aligned} \partial^\alpha u_k(x) - \partial^\alpha u(x) &= \frac{1}{2\pi R} \partial_x^\alpha \int_{|y|=R} P_R(x-a,y)(u_k(a+y) - u(a+y))\, ds(y) \\ &= \frac{1}{2\pi R} \int_{|y|=R} \partial_x^\alpha P_R(x-a,y)(u_k(a+y) - u(a+y))\, ds(y). \end{aligned}$$

Here, $|\partial_x^\alpha P_R(x-a,y)| \leq C_{\alpha,R}$ for any $|y| = R$ and $|x-a| \leq R/2$. Therefore,

$$|\partial^\alpha u_k - \partial^\alpha u| \leq C_{\alpha,R} \max_{|y|=R} |u(a+y) - u_j(a+y)| \to 0.$$

Covering a compact set $K \subseteq \Omega$ by finitely many open discs of this form $|x-a| \leq R/2$ for $R = R(a) > 0$, we get that $\partial^\alpha u_k \to \partial^\alpha u$ uniformly on K. $\square$

1.2 Upper semicontinuous functions

Definition 1.1. Let X be a metric space. A function $u : X \to [-\infty, \infty)$ is called **upper semicontinuous** if for every $a \in \mathbb{R}$, the set $\{x \in X : u(x) < a\}$ is open.

Proposition 1.1. *A function $u : X \to [-\infty, \infty)$ is upper semicontinuous if and only if $\limsup_{y \to x} u(y) \leq u(x)$ for all $x \in X$.*

Example 1.1. Let $F \subseteq X$ is closed. Then $\mathbb{1}_F$ is upper semicontinuous.

Proposition 1.2. *If u is upper semicontinuous, and $K \subseteq X$ is compact, then u is bounded above, and $\sup_K u$ is achieved.*

Proposition 1.3. *Let $u : X \to [-\infty, \infty)$ be upper semicontinuous and bounded above. Then there exists a sequence $u_j \in C(X)$ such that $u_1 \geq u_2 \geq \cdots \geq u$ and $u_j \to u$ pointwise.*

Example 1.2. Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in \mathrm{Hol}(\Omega)$. Then $u = \log|f|$ (with $\log(0) = -\infty$) is upper semicontinuous.

1.3 Subharmonic functions

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^2$ be open. We say that a function $u : \Omega \to [-\infty, \infty)$ is **subharmonic** if

1. u is upper semicontinuous.
2. If $K \subseteq \Omega$ is compact and $h \in C(K) \cap H(K^o)$ is such that $u \leq h$ on ∂K, then $u \leq h$ on K.

Example 1.3. If u is harmonic, then by the mean value property, u is subharmonic.

Proposition 1.4. *Let $(u_\alpha)_{\alpha \in A}$ be a family of subharmonic functions on Ω such that $u = \sup_\alpha u_\alpha < \infty$ pointwise and u is upper semicontinuous. Then u is subharmonic. If (u_j) is a decreasing sequence of subharmonic functions, then $u = \lim u_j$ is subharmonic.*

Proof. The first statement is immediate from the definition. For the second statement, first note that that $u = \lim u_j = \inf u_j$ is upper semicontinuous (if u_α is upper semicontinuous for each α, then $\inf_\alpha u_\alpha$ is, as well).

Now let $K \subseteq \Omega$ be compact, let $h \in C(K) \cap H(K^o)$, and let $u \leq h$ on ∂K. Let $\varepsilon > 0$, and let $x_0 \in \partial K$. Then there exists a j such that $u_j(x_0) < u(x_0) + \varepsilon \leq h(x_0) + \varepsilon$. Then $(u_j - h)(x_0)$, where $u_j - h$ is upper semicontinuous on K. So there is a neighborhood V_{x_0} of x_0 such that $u_j(x) - h(x) < \varepsilon$ for all $x \in V_{x_0} \cap \partial K$. Then, for all $k \geq j$, $u_k(x) - h(x) < \varepsilon$ for all $x \in V_{x_0} \cap \partial K$. Covering the compact set ∂K by finitely many open sets of the form V_{x_0}, we get $u_j \leq h + \varepsilon$ on ∂K for all large j. By the subharmonicity of the u_j, we get that $u_j \leq h + \varepsilon$ on K, so $u \leq h$ on K. $\square$

Remark 1.1. This is the same argument as in the standard proof of Dini's theorem in elementary analysis.

Theorem 1.2. *Let $u : \Omega \to [-\infty, \infty)$ be upper semicontinuous. The following are equivalent:*

1. *u is subharmonic*

2. *(local sub-mean value inequality): For every $a \in \Omega$,*

$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y)$$

   for all small $R > 0$.

3. *For every $a \in \Omega$,*

$$u(a) \leq \frac{1}{\pi R^2} \iint_{|y|\leq R} u(a+y)\, dy$$

   for all small $R > 0$, where dy is Lebesgue measure in $\mathbb{R}^2$.

We will prove these, along with more equivalences, next time.